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LIMIT THEOREMS FOR SUMS OF INDEPENDENT VARIABLES
TAKING INTO ACCOUNT LARGE DEVIATIONS. II

By

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LIMIT THEOREMS FOR SUMS OF INDEPENDENT VARIABLES TAKING INTO ACCOUNT LARGE DEVIATIONS. II

Yu. V. Linnik

2. Narrow zones of local and integral normal attraction.

0. Let us follow the symbols used in Part I [16]. Let us consider the narrow zones of local normal convergence for a variable X_j of class (d) [16]. It will be more convenient for us to start not from the zones $[0, \psi(n)]$ and $[h - \psi(n), 0]$, but from the functions $h(x)$, which were described in Section 1 of Part I [16]. Let us examine the conditions

$$\mathbf{E} \exp h(|X_j|) < \infty. \tag{0.1}$$

Let the function $h(x)$ of class I be given [16]. It is subject to the conditions [16]:

$$(\log x)^{2+\zeta_0} \leqslant h(x) \leqslant x^{1/2}, \quad (x \geqslant 1) \tag{0.2}$$

$h(x) = \exp(H(\log x))$, where $H(x)$ is monotonic and differentiable

$$H'(z) \leqslant 1; \; H'(z) \to 0 \text{ [redacted] } z \to \infty, \tag{0.3}$$

$$H'(z) \exp H(z) > c_1 z^{1+\zeta_1}. \tag{0.4}$$

Let us define a new positive function $\Lambda(n)$ by the equation

$$h(\sqrt{n}\Lambda(n)) = (\Lambda(n))^2. \tag{0.5}$$

Theorem 1. Condition (0.1), where $h(x)$ belong to class I, is necessary in order that the zones $[0, \Lambda(n)\rho(n), 0]$ be z. u. l. n. a. [zones of uniform linear normal attraction], and is sufficient in order that the zones $[0, \Lambda(n)/\rho(n)]$; $[-\Lambda(n)/\rho(n), 0]$ be z. u. l. n. a.

Theorem 2. The same proposition as in Theorem 1 is valid for z. n. a. (integral), and convergence in the corresponding zones is uniform.

For $h(x)$ belonging to class II, let us define $\Lambda(n)$ from the condition: $\Lambda(n) = \sqrt{h(n)} = \sqrt{M(n)\log n}$. Because of the properties of $h(n)$, the defined $\Lambda(n)$ will differ from $\rho(n)$ only by the factor of $\Lambda(n)$, which was determined from Eq. (0.5). Further, the following theorems are valid:

Theorem 3. If $h(x)$ belongs to class II, then a proposition analogous to Theorem 1 for the above function $\Lambda(n)$ is valid for the z. u. l. n. a.

Theorem 4. For $h(x)$ of class II, and the above-indicated $\Lambda(n)$, a proposition analogous to Theorem 3 is valid for the z. n. a.

For $h(x)$ of class III we have the narrowest zones. Here let $\Lambda(n) = \sqrt{\log n}$.

Theorem 5. For $h(x)$ of class III and $\Lambda(n) = \sqrt{\log n}$, the same proposition as in Theorem 1 is valid for the z. u. l. n. a. and the variables $X_j \in (d)$; the same proposition as in Theorem 2 is valid for the z. n. a.

1. First of all, let us explain the appropriateness of conditions (0.2), (0.3) and (0.4) imposed on $h(x)$.

The condition $h(x) \geq (\log x)^{2+\xi_0}$ (the significance of which is to be explained later) ensures that the zones are not the narrowest, and it can be compared with the condition $E \exp 3 \log|X_j| < \infty$, which ensures only the existence of a third moment. The condition $h(x) \leq$

$\leq x^{1/2}$ indicates that the zones are narrow [16]. According to Theorem 3 of the previous work [16], a condition of the form of (2.1) of this work [16] corresponds to the zones $[0, n^{\alpha}(\rho(n))^{\pm 1}]$; $[-n^{\alpha}(\rho(n))^{\pm 1}, 0]$ when $\alpha < \frac{1}{6}$. At $\alpha < \frac{1}{6}$, the exponent of $4\alpha/2\alpha + 1$ under this condition is $< 1/2$. It is natural to assume that $h(x)$ is monotonic and differentiable, and to represent it in the form of $h(x) = \exp(H(\log x))$ when $x \geq 1$, where $H(z)$ is the same. If it is assumed that $h'(x)$ is monotonic (which is natural), then this leads to condition (0.3) (the narrowness of the zone). Violation of (0.4) with the indicated monotonicity of $h'(x)$ would lead to violation of the left inequality in (0.2).

2. Let us prove that condition (0.1) is necessary in order that $[0, \Lambda(n)\rho(n)]$; $[-\Lambda(n)\rho(n), 0]$ be z. n. a. Let these zones be z. n. a. Then at $n > n_0$ we have (because $\Lambda(n) \to \infty$ as $n \to \infty$)

$$P\left(Z_n > \frac{\Lambda(n)\rho(n)}{2}\right) < \exp\left(-\frac{(\Lambda(n)\rho(n))^2}{16}\right). \qquad (2.1)$$

Let condition (0.1) not be fulfilled. Then a sequence $x_m \to \infty$ is found such that either

$$P(X_1 > x_m) > \exp(-2h(x_m)), \qquad (2.2)$$

or

$$P(X_1 < -x_m) > \exp(-2h(|x_m|)). \qquad (2.3)$$

Let (2.2) be fulfilled. For a sufficiently large value of m, let us take n such that $x_m = \sigma n^{1/2}\Lambda(n)\rho(n) + \theta$; $|\theta| \leqslant 1$. The event indicated in (2.1) certainly occurs if two independent events occur: $X_1 > \sigma n^{1/2}\Lambda(n)\rho(n) + \theta, X_2 + X_3 + \ldots + X_n/\sigma\sqrt{n}| < 1$. Hence, according to the central limit theorem and (2.2), we have:

$$P\left(Z_n > \frac{\Lambda(n)\rho(n)}{2}\right) > c_0 P(X_1 > x_m) > c_0 \exp(-2h(x_m)) = \qquad (2.4)$$
$$= c_0 \exp[-2h(\sigma\sqrt{n}\Lambda(n)\rho(n) + 0)] > c_0 \exp[-2h(2\sigma\sqrt{n}\Lambda(n)\rho(n))].$$

When ξ, η are sufficiently large, according to condition (0.3)

$$h(\xi\eta) = \exp H(\log\xi + \log\eta) \leqslant \exp(H(\log\xi) + o(\log\eta)) = h(\xi)\,\eta^{o(1)}. \qquad (2.5)$$

Letting $\xi = \sqrt{n}\Lambda(n);\ \eta = 2\sigma\rho(n)$, we find

$$2h(2\sigma\sqrt{n}\Lambda(n)\rho(n)) = 2(\Lambda(n))^2(\rho(n))^{o(1)}, \qquad (2.6)$$

which contradicts (2.1). Analogous reasoning is carried out for case (2.3). If $X_j \in (d)$ [1], then condition (0.1) is necessary in order that $[0, \Lambda(n)\rho(n)]$ and $[-\Lambda(n)\rho(n), 0]$ both be z. u. l. n. a. The proof is analogous to the preceding one, taking into account the reasoning in Section 4 of Part I [16].

3. Let us consider the problem of whether condition (0.1) is sufficient in order that the zones $[0, \Lambda(n)\rho(m)]$ and $[-\Lambda(n)\rho(n), 0]$ be z. u. l. n. a. when $X_j \in (d)$. In condition (0.1) we can examine the function $ah(x)$ instead of $h(x)$ (positive constants), and, therefore, without losing generality, assume $\sigma = 1$. At a given $\rho(n)$, let us define the function $\Lambda_\rho(n)$ from the equation

$$h\left(\Lambda_\rho(h)\frac{\sqrt{n}}{\rho(n)}\right) = (\Lambda_\rho(n))^2. \qquad (3.1)$$

If we prove that the zone $[0, \Lambda_\rho(n)]$ is a z. u. l. n. a., then the zone $[0, \frac{\Lambda(n)}{\rho(n)}]$ will also be, since it is narrower. Indeed, letting $\Lambda_\rho(n) = \Lambda(n)\gamma(n)$, we find from (3.1), (0.5) and the reasoning in Section 2

$$(\gamma(n)(\rho(n))^{-1})^{o(1)} \quad (\gamma(n))^2;\ \gamma(n) = \left(\frac{1}{\rho(n)}\right)^{o(1)}.$$

Let

$$\log\Lambda_\rho(n) = X_\rho(n), \qquad (3.2)$$

and we obtain from (3.1)

$$H\left(\log\Lambda_\rho(n) + \log\frac{\sqrt{n}}{\rho(n)}\right) = 2X_\rho(n). \qquad (3.3)$$

Now letting

$$l_\rho(n) = \log \frac{\sqrt{n}}{\rho(n)} = \frac{1}{2}\log n - \rho_1(n). \tag{3.4}$$

we have

$$H(X_\rho(n) + l_\rho(n)) = 2X_\rho(n);\ X_\rho(n) + l_\rho(n) = H^{-1}(2X_\rho(n)). \tag{3.5}$$

Further, let us determine the small number μ from the condition

$$n^{1-2\mu} = e^B(\Lambda_\rho(n))^2 = e^B \exp(2X_\rho(n)) \tag{3.6}$$

(B, as in Part I [16], is a bounded number, not always the same).

Inasmuch as

$$(1-2\mu)\log n = B + 2X_\rho(n),$$

$$\mu \log n = B + \frac{\log n}{2} - X_\rho(n) = l_\rho(n) + X_\rho(n) + \rho_2(n) - 2X\ (n), \tag{3.7}$$

taking (3.5) into account, we find

$$\mu \log n = H^{-1}(2X_\rho(n)) + \rho_2(n) - 2X_\rho(n), \tag{3.8}$$

while (3.7) gives

$$\mu = \frac{1}{2} - \frac{X_\rho(n)}{\log n} + \frac{B}{\log n}. \tag{3.9}$$

4. Now, in the symbols of Part I [16], we obtain formulas (5.6) and (6.5):

$$p_{z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-n^{-\mu}}^{n^{-\mu}} (\varphi(t))^n \exp(-itx\sqrt{n})\,dt + B\exp(-c_1 e^{2X_\rho(n)}). \tag{4.1}$$

Bearing in mind the use of the method [16], we wish to bound $\varphi^{(q)}(t)$ when $|t| \leq n^{-\mu}$:

$$\varphi^{(q)}(t) = B\int_1^\infty x^q \exp(-h(x))\,dx = B\int_1^\infty \exp(q\log x - h(x))\,dx. \tag{4.2}$$

Now let us establish that

$$xh'(x) \to \infty \text{ as } x \to \infty. \tag{4.3}$$

In fact

$$h(x) = \exp H(\log x);\ xh'(x) = \frac{dh(x)}{d\log x} = \exp(H(\log x))H'(\log x) \to \infty$$
$$\text{as } x \to \infty$$

on the strength of (0.4). The integrand in (4.2) in the exponent has the form of $q \log x - h(x)$, and its derivative is $\frac{q}{x} - h'(x)$.

Let us set up the saddle-point equation

$$xh'(x) = q. \tag{4.4}$$

On the strength of (4.3), it has a unique solution for $q > q_0$. Let us denote this solution by $Q_0(q)$.

Lemma 1. *A bound of the following form is valid:*

$$\int_1^\infty x^q \exp(-h(x))\,dx = B\exp(Bq + q\log Q_0(q) - h(Q_0(q))). \tag{4.5}$$

Let us find the abscissa $x = x_1$ such that when $x \geq x_1$

$$h(x) \geqslant 2q\log x;\ h(x) - q\log x \geqslant \frac{h(x)}{2} \tag{4.6}$$

Since $h(x) \geq (\log \alpha)^{2+\xi_0}$ according to (0.2), then (4.6), certainly, will be fulfilled if $(\log x)^{2+\xi_0} \geqslant 2q\log x;\ \log x \geqslant (2q)^{\frac{1}{1+\xi_0}};\ x \geqslant \exp(2q)^{\frac{1}{1+\xi_0}} = x_1,$ since when $x \geqslant x_1$ $h(x) - q\log x \geqslant \frac{1}{2}h(x) \geqslant \frac{1}{2}(\log x)^{2+\xi_0},$ we have

$$\int_{x_1}^\infty x^q \exp(-h(x))\,dx = B. \tag{4.7}$$

The function $q \log x - h(x)$ has a maximum at $x = Q_0(q)$ and then decreases (see (4.4)) so that

$$\int_1^\infty x^q \exp(-h(x))\,dx = B\exp(Bq + q\log Q_0(q) - h_1(Q_0(q))). \tag{4.8}$$

Using Lemma 1, we obtain

$$\frac{\varphi^{(q)}(t)}{q!} = B\exp q\left(B + \log Q_0(q) - \frac{1}{q}h(Q_0(q)) - \log q\right). \tag{4.9}$$

Letting (as in Part I [16], (6.7))

$$\tilde{\varphi}_q(t + t_0) = \varphi(t_0) + \frac{t\varphi'(t_0)}{1!} + \ldots + \frac{t^q\varphi^{(q)}(t_0)}{q!}, \tag{4.10}$$

we obtain when $K(t) = \log \varphi(t)$, $(|t| \leq \varepsilon_0)$:

$$K^{(q)}(t_0) = (\log \tilde{\varphi}_q(t + t_0))^q|_{t=0}. \tag{4.11}$$

5. Let

$$L(q) = \log Q_0(q) - \frac{1}{q}h(Q(q)) - \log q \tag{5.1}$$

and let us show that L(q) is a non-decreasing function of $\underline{q}$. From the equality

$$Q_0(q)\,h'(Q_0(q)) = q \tag{5.2}$$

by simple calculation it is deduced that

$$L'(q) = \frac{1}{q}\left(\frac{1}{q}h(Q_0(q)) - 1\right). \tag{5.3}$$

Now it is sufficient to show that $h(Q_0(q)) \geqq q$ or $h(Q_0(q)) \geqq Q_0(q) \cdot h'(Q_0(q))$ or $d \log h(y)/d \log y \leqq 1$, i.e., $H'(z) \leqq 1$, which corresponds to (0.3). Thus, L(q) is monotonic. From (4.9) let us derive

$$\frac{\varphi^{(q)}(t_0)}{q!}t^q = B|t|^q \exp(Bq + qL(q)). \tag{5.4}$$

Let us choose the contour $|t| = e^{-\nu}$ such that (5.4) is sufficiently small over it. Let

$$\nu = \nu(q) = c_1 + L(q), \tag{5.5}$$

where c_1 is sufficiently large. Then, when $|t| = e^{-\nu}$

$$B\exp(Bq - c_1 q - qL(q) + qL(q)) = B\exp\left(-\frac{c_1}{2}q\right) < \left(\frac{1}{4}\right)^q, \tag{5.6}$$

$$\frac{t\varphi'(t_0)}{1!} + \frac{t^2\varphi''(t_0)}{2!} + \ldots + \frac{t^q\varphi^{(q)}(t_0)}{q!} < \frac{1}{2}, \tag{5.7}$$

so that

$$\frac{3}{2} > |\tilde{\varphi}_q(t)| > \frac{1}{2}. \tag{5.8}$$

Hence when $|t_0| \leq n^{-\mu}$

$$K^{(q)}(t_0) = \frac{r!}{2\pi i}\oint_{|t|=e^{-\nu}} \log\tilde{\varphi}_q(t)\frac{dt}{t^{q+1}} = Bq'\oint_{|t|=e^{-\nu}}\frac{|dt|}{|t|^{q+1}} = \tag{5.9}$$

$$= Bq'\exp(q^{\nu}(q)) = B\exp(Bq + q\log Q_0(q) - h(Q_0(q))).$$

$$\sup_{|t|<n^{-\mu}}\left|K^{(m)}(t)\frac{t^m}{m!}\right| = B\exp\Big\{m\big(B + \log Q_0(m) - \tag{5.10}$$

$$-\frac{1}{m}h(Q_0(m)) - \log m - \mu\log n\big)\Big\}.$$

Now let us determine $\underline{m}$ by using the equalities

$$\log Q_0(m) = H^{-1}(2X_p(n)); \tag{5.11}$$

$$\exp H(\log Q_0(m)) = h(Q_0(m)) = \exp[2X_p(n)]. \tag{5.12}$$

Let us substitute this into (5.10) and, taking (3.8) into account, reduce (5.10) to the form

$$B\exp\left\{m\left(B-\rho_2(n)+2X_\rho(n)-\log m-\frac{1}{m}\exp 2X_\rho(n)\right)\right\}=$$
$$=B\exp(m(B-\rho_2(n)-\exp(2X_\rho(n)-\log m)+2X_\rho(n)-\log m)). \tag{5.13}$$

Now let us show that

$$2X_\rho(n)-\log m\to\infty \quad \text{as} \quad n\to\infty. \tag{5.14}$$

On the strength of (5.2)

$$\exp H(\log Q_0(m))H'(\log\varphi_0(m))=Q_0(m)h'(Q_0(m))=m. \tag{5.15}$$

From (5.11) we find: $H(\log\varphi_0(m))=2X_\rho(n)$. Let us substitute into (5.15)

$$\exp(2X_\rho(n))H'(\log Q_0(m))=\exp(2X_\rho(n))H'(X_\rho(n)+l\ (n))=m.$$

Hence, on the strength of (0.3),

$$2X_\rho(n)-\log n=-\log H'(X_\rho(n)+l_\rho(n))\to\infty. \tag{5.16}$$

Using (5.14), we obtain for (5.13) the bound

$$B\exp(-c_2\exp(2X_\rho(n))). \tag{5.17}$$

Note that because of (3.6), $\exp(2X_\rho(n))=Bn^{1-2\mu}$.

6. Now (5.10) gives

$$\sup_{|t|<n^{-\mu}}\left|K^{(m)}(t)\frac{t^m}{m!}\right|=B\exp(-c_2\exp(2X_\rho(n))), \tag{6.1}$$

so that when $|t|\le n^{-\mu}$

$$K(t)=-\frac{t^2}{2}+\sum_{r=3}^{m}\psi_r\frac{t^r}{r!}+B\exp(-c_2\exp 2X_\rho(n)), \tag{6.2}$$

where

$$\psi_r=K^{(r)}(0)=B\exp(Br+r\log Q(r)-h(Q_0(r))). \tag{6.3}$$

In view of the fact that $2X_\rho(n)>H\left(\log\frac{\sqrt{n}}{\rho(n)}\right)=\log h\left(\frac{\sqrt{n}}{\rho(n)}\right)>\exp(2X_\rho(n))>$ $>h\left(\frac{\sqrt{n}}{\rho(n)}\right)>\left(\log\frac{\sqrt{n}}{\rho(n)}\right)^{s+l_0}$, $n\exp(-c_2\exp 2X_\rho(n))=B\exp(-c_3\exp 2X_\rho(n))$, so that

$$nK(t)=nK_s(t)+B\exp(-c_4n^{1-2\mu}), \tag{6.4}$$

where

$$K_3(t) = -\frac{t^2}{2} + \sum_{r=3}^{m} \psi_2 \frac{t^2}{r!}. \tag{6.5}$$

Hence it is easy to derive

$$p_{z_n}(x) = \frac{\sqrt{n}}{2\pi} \int_{-n^{-\mu}}^{n^{-\mu}} \exp\left(-\frac{nt^2}{2} + K_3(t) - i\sqrt{n}tx\right) dt + B\exp(-c_4 n^{1-2\mu}), \tag{6.6}$$

where

$$K_3(t) = \sum_{r=3}^{m} \psi_r \frac{t^r}{r!}. \tag{6.7}$$

Let us take the integral function

$$\exp(nK_3(t)). \tag{6.8}$$

Let $\mu_1 = \mu - \omega_n$, where the small number ω_n is determined as follows: if $\exp(2X_\rho(n))/m \geq \log n$, then we take $\omega_n = \frac{1}{100}$, and if $\exp 2X_\rho(n)/m \geq \log n$, we take $\omega_n = \frac{1}{100} \frac{\exp 2X_\rho(n)}{m \log n}$.

When $r \leq m$ and $|t| \leq n^{-\mu_1}$, let us consider

$$\frac{\psi_r t^r}{r!} = B \exp r\left(B + \log\varphi(r) - \frac{1}{r} h(Q_0(r)) - \log r - \mu_1 \log n\right). \tag{6.9}$$

Now we note that $X_\rho(n) = o(\log n)$, as is apparent from (3.3) and (0.3) (because according to (0.3), $H(z) = o(z)$). In view of this, it follows from (3.7) that

$$\mu = \frac{1}{2} + o(1),\ \mu \geqslant 0{,}49 + o(1). \tag{6.10}$$

Therefore, when $3 \leq r < c_3$

$$n\frac{\psi_r t^r}{r!} = o(1). \tag{6.11}$$

Let us move to the case of $c_3 \leq r \leq m$. Let us isolate the values $c_3 \leq r \leq (\log n)1 + \xi/2$ and prove the bound

$$\log Q_0(r) = B \exp r^{\frac{1}{1+\xi_1}}. \tag{6.12}$$

From (0.4) we have $xh'(x) > (\log x)^{1+\xi_1}$. In view of this, from the equality $Q_0(r)h'(Q_0(r)) = r$ it follows that $(\log Q_0(r))^{1+\xi_1} < r;\ Q_0(r) < \exp r^{\frac{1}{1+\xi_1}}$, which proves (6.12). At the indicated values of r, (6.9) gives

$$B\exp r\left(B + r^{\frac{1}{1+\xi_1}} - 0{,}48\log n\right) = Bn^{-0{,}4r}. \tag{6.13}$$

Summation of these expressions with respect to $c_3 \leqslant r \leqslant (\log n)^{1+\xi_1/2}$ gives

$$Bn^{-2}. \tag{6.14}$$

Now let $(\log n)^{1 + \xi_1/2} < r \leq m$. Because of the monotonicity of the function $L(r)$ (see Section 5), (6.9) does not exceed

$$\left(B + \log Q_0(m) - \frac{1}{m} h(Q_0(m)) - \log m - \mu_1 \log n\right). \tag{6.15}$$

By comparison with (5.10) and (5.13), we obtain the bound for (6.15)

$$B \exp r \left(B - \frac{1}{2m} \exp 2X_{f}(n) - \rho_2(n) + \omega_n \log n\right). \tag{6.16}$$

Hence taking the definition of the number ω_n into account, we find for (6.9)

$$B \exp r \left(-\frac{\rho_2(n)}{2} - \frac{1}{8} \log n\right) + B \exp r \left(-\frac{\rho_2(n)}{2} - \frac{1}{8n} \exp 2X(n)\right). \tag{6.17}$$

By summing these expressions with respect to r, $(\log n)^{1+\xi_1/2} \leqslant r \leqslant m$, for the corresponding part of $K_3(t)$ we obtain the bound

$$Bn^{-2}. \tag{6.18}$$

Gathering together bounds (6.11), (6.14) and (6.18), we find, when $|t| = n^{-\mu_1}$,

$$nK_3(t) = B \exp nK_3(t) = B. \tag{6.19}$$

7. Let

$$\exp nK_3(t) = \sum_{r=3}^{\infty} \chi_r \frac{t^r}{r!}, \tag{7.1}$$

then, in accordance with (6.19),

$$\chi_r = \frac{r!}{2\pi i} \oint_{|t|=n^{-\mu_1}} \frac{\exp(nK_3(t))}{t^{r+1}} dt = Br!\ n^{r\mu_1}, \tag{7.2}$$

$$\frac{\chi_r}{r!} t^r = Bn^{r(\mu_1-\mu)} = B \exp(-2\omega_n \log n), \text{ при } r \geqslant m,\ |t| \leqslant n^{-\mu}. \tag{7.3}$$

The numbers ω_n have various values depending upon the behavior of $\exp(2X_\rho(n))/m$, as described in Section 6. Each of these cases reduces to the bound

$$B \sum_{r=m}^{\infty} \exp\left(-\frac{r}{100m} \exp[2X_\rho(n)]\right) = B \exp\left(-\frac{1}{200} \exp[2X_\rho(n)]\right). \tag{7.4}$$

Thus
$$\exp[nK_3(t)] = 1 + \sum_{r=3}^{m} \frac{\chi_r t^r}{r!} + B\exp(-c_4 \exp[2X_p(n)]),$$
$$p_{Z_n}(x) = \frac{\sqrt{n}}{2\pi} \int_{-n^{-\mu}}^{n^{-\mu}} \exp\left(-\frac{nt^2}{2} - itx\sqrt{n}\right)\left(1 + \sum_{r=3}^{m} \frac{\chi_r t^r}{r!}\right) + \tag{7.5}$$
$$+ B\exp(-c_5 e^{2X_p(n)}), \tag{7.6}$$
$$p_{Z_n}(x) = \frac{1}{2\pi} \int_{-n^{1/2}-\mu}^{n^{1/2}-\mu} \exp\left(-\frac{\xi^2}{2} - i\xi x\right)\left(1 + \sum_{r=3}^{m} \frac{\chi_r}{r!}\left(\frac{\xi}{\sqrt{n}}\right)^r\right) +$$
$$+ B\exp(-c_6 e^{2X_p(n)}). \tag{7.7}$$

Now let us consider
$$\int_{n^{1/2}-\mu}^{\infty} e^{-\xi^2/2} \frac{\chi_r}{r!}\left(\frac{1}{\sqrt{n}}\right)^r \xi^r d\xi. \tag{7.8}$$

Following Section 8 of Part I [16], we obtain for (7.8) the bound, when $r \leq m$,
$$B\exp\left(-\frac{1}{4}n^{1-2\mu}\right)\exp r\left(B + \frac{\log r}{2} - \left(\frac{1}{2} - \mu_1\right)\log n\right) =$$
$$= B\exp\left(-\frac{1}{4}n^{1-2\mu}\right)\exp r\left(B + \frac{\log m}{2} - \left(\frac{1}{2} - \mu_1\right)\log n\right). \tag{7.9}$$

Further, since $\exp(2X_p(n)) = Bn^{1-2\mu};\ X_p(n) = \left(\frac{1}{2} - \mu\right)\log n + B,$

then
$$\frac{1}{2}\log m - \left(\frac{1}{2} - \mu_1\right)\log n = \frac{1}{2}\log m - \left(\frac{1}{2} - \mu\right)\log n + (\mu_1 - \mu)\log n =$$
$$= \frac{1}{2}\log m - X_p(n) + B - \omega_n < \frac{1}{2}\log m - X_p(n) \to -\infty.$$

By summing the bounds of (7.9) with respect to r = 3, 4, ..., m, we find the error
$$B\exp\left(-\frac{1}{4}n^{1-2\mu}\right) = B\exp(-c_4 \exp 2X_p(n)). \tag{7.10}$$

By bounding the integral over $(-\infty, -n^{1/2} -\mu)$ in the same way, we arrive at the formula
$$p_{Z_n}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp\left(-\frac{\xi^2}{2} - i\xi x\right)\left(1 + \sum_{r=3}^{m} \frac{\chi_r^{-1}}{r!}\left(\frac{\xi}{\sqrt{n}}\right)^r\right) +$$
$$+ B\exp(-c_5 \exp(2X_p(n))). \tag{7.11}$$

8. At $3 \leq r \leq n$, let us now examine [16]
$$\int_{-\infty}^{\infty} \exp(-\xi^2/2 - i\xi x)\,\xi^r d\xi = H_r^{(0)}(x)\,e^{-x^2/2}. \tag{8.1}$$

At $3 \leq r \leq c_3$ we have for (8.1) the bound

$$Be^{-x^2/2}\sum_{r=3}^{c_3}\frac{|\chi_r|}{r!}|x|^r\frac{1}{(\sqrt{n})^r}. \tag{8.2}$$

When

$$|x|\leqslant\frac{n^{1/2-\mu}}{\rho(n)}. \tag{8.3}$$

(8.1) has the bound

$$Be^{-x^2/2}\sum_{r=3}^{c_3}n^{r\mu_1}n^{\left(\frac{1}{2}-\mu\right)}n^{-r/2}\frac{1}{(\rho_1(n))^3}=Be^{-x^2/2}\frac{1}{(\rho_1(n))^3}. \tag{8.4}$$

Now let $c_3 < r \leq m$. Let us assume $r = q$; $s = \rho q (0 \leq \rho \leq 1/2)$. Following Section 9 (see (9.7) of Part I [16]), we obtain for $c_4 \leq q \leq m$

$$\frac{\chi_q}{q!}n^{-q/2}H_q^{(s)}(x)=B\sup_{\rho}q\,(B-(1-2\rho)\log\rho_1(n)+ \\ +\rho\log q-\rho(1-2\mu)\log n+(\mu_1-\mu)\log n). \tag{8.5}$$

Since $\exp(2X_e(n))/m\to\infty$, then $B_n^{1-2\mu}/m\to\infty$, hence $\log q<\log m\leqslant B+(1-2\mu)\log n$. Thus $\rho\log q-\rho(1-2\mu)\log n=B-v(n)$, where $v(n) \geq 0$. Then (see Section 6) $(\mu_1-\mu)\log n=-\omega_n$, $\log n\to-\infty$, as $n\to\infty$. Thus (8.5) obtains the bound

$$B\exp(-c_4 q). \tag{8.6}$$

If $q \geq \rho_2(n)$, then the sum of these bounds at $\rho_2(n) < q < m$ is added to the total bound

$$\frac{B}{\rho_3(n)}. \tag{8.7}$$

Then let $\rho_2(n)$ be chosen such that

$$\log q\leqslant\frac{(1-2\mu)\log n}{2}=\left(\frac{1}{2}-\mu\right)\log n \tag{8.8}$$

(note that $(1 - 2\mu) \log n \to \infty$). Then (8.5) has the bound

$$\sup_{\rho}\left(B\exp(-q(1-2\rho)\log\rho_1(n)\right)+B\exp\left(-q\rho\left(\frac{1}{2}-\mu\right)\log n\right)\Big)=B\exp(-q\rho_3(n)). \tag{8.9}$$

By summing the latter with respect to $3 \leq q \leq \rho_2(n)$, we obtain the bound

$$\frac{B}{\rho_4(n)}. \tag{8.10}$$

This proves the local limit theorem for the zones $[0, n^{1/r-\mu}/\rho_1(n)]$ and $[-n^{1/r-\mu}/\rho_1(n), 0]$. Then, on the strength of (3.1) and (3.6), $\Lambda_\rho(n)=\exp(X_\rho(n))=$

$= e^B n^{1/2-\mu}$. This proves Theorem 1.

9. Let us consider integral Theorem 2, following Sections 11 through 14 of Part I [16]. Now let X_j have random values of a general form; $\mathbf{E}X_j = 0$; $\mathbf{D}(X_j) = 1$. Let h(x) be a class-I function and let condition (0.1) be fulfilled. At first let us follow Section 3. Let us define μ from condition (3.6) and let $\alpha = \frac{1}{2} - \mu$. We shall prove that the zones $[0, n^{\alpha}/\rho_5(n)]$ and $[-n^{\alpha}/\rho_5(n), 0]$ are z. n. a. Let us go into the first of these. We shall follow the symbols and reasoning of Sections 11 through 14 of Part I [16], pointing out only the essential differences. It is important for us that (see (3.3))

$$n^{1/2-\mu} = n^{\alpha} = e^B \exp(X_\rho(n)) > \left(h\left(\frac{\sqrt{n}}{\rho(n)}\right)\right)^{1/2} > (\log n)^{1+\varepsilon_0/2}. \tag{9.1}$$

Following the reasoning and symbols of Sections 11 through 14 of Part I [16], we introduce Y_n, $\psi_n(t)$ and arrive at formula (12.3) of that work.

Note that $n^{2\alpha} = n^{1-2\mu} > (\log n)^{2+\varepsilon_0}$, so that

$$n^{K+1} \exp(-c_6 n^{2\alpha}) = \exp(-c_7 n^{2\alpha}) \tag{9.2}$$

(see formula (12.5) of Part I [16]).

In view of this, formulas (12.7) and (12.8) and (13.1) through (13.5) of Part I [16] are valid. Bound (7.2) of this work is taken for χ_r. Following Section 13 [16], we arrive at formula (13.11). We must, however, then give a more accurate bound in this formula. It is written as follows:

$$\int_{x_1}^{\infty} e^{-x^2/2} H_q^{(0)}(x)\,dx = B^q q! \sum_{s<\frac{q}{2}} \frac{1}{q!\,(q-2s)!} \int_{x_1}^{\infty} e^{-u^2/2} u^{q-2s}\,du. \tag{9.3}$$

When $1 \leqslant x_1 \leqslant n^{\alpha}/\rho_6(n)$, $Q = q - 2s = q(1-2\rho)$ the bound is easily found by the saddle-point method ($Q \geq 1$):

$$\int_{x_1}^{\infty} e^{-u^2/2} u^Q\,du = B^Q x_1^{Q-1} \exp\left(-\frac{x_1^2}{2} + \frac{1}{2}Q\left|\log\frac{Q}{x^2}\right| + \frac{1}{2}Q\right). \tag{9.4}$$

We see that if $x_1^2 > m > Q;\ x_1 \geqslant \sqrt{m} > \sqrt{Q}$, then (9.4) reduces to the bound

$$B^Q x_1^{Q-1} \exp\left(-\frac{x_1^2}{2}\right) \tag{9.5}$$

Reasoning as in Section 8, we arrive at a proof of the integral limit theorem for the zone $[\sqrt{m},\ n^{\alpha}/\rho_5(n)]$. Now let $x_1 < \sqrt{m}$. Then (9.4) is bounded by the value

$$B^Q x_1^{Q-1} \exp\left(-\frac{x_1^2}{2}\right) \exp\left(\frac{1}{2} Q \log Q\right). \tag{9.6}$$

When $s = \rho q,\ 0 < \rho < \frac{1}{2}$ we take the corresponding term of (9.3), taking into account bound (9.9) and the bounds $\chi_q/q! = B \exp(\mu_1 q \log n)$; $n^{-q/2} = \exp(-q \log n/2)$, which must be multiplied by that term. As a result, when multiplying by $x_1 \exp\left(-\frac{x_1^2}{2}\right)$ we have the following bound (see Section 9 [16]; $x_1 \leq \sqrt{m}$):

$$\begin{aligned} & B \exp q \left(B + (1-2\rho)\frac{1}{2}\log m - \rho \log q - (1-2\rho)\log q + \log q - \right. \\ & \qquad \left. - \rho \log \rho - (1-2\rho)\log(1-2\rho) - \frac{1}{2}\log n + \mu_1 \log n\right) = \\ & = B \exp q \left(B + \left(\frac{1}{2} - \rho\right)\log m + \rho \log q - \rho \log \rho - (1-2\rho)\log(1-2\rho) - \right. \\ & \left. - \left(\frac{1}{2} - \mu_1\right)\log n\right) = B \exp q \left(B + \frac{1}{2}\log m - \left(\frac{1}{2} - \mu\right)\log n - \omega_n \log n\right). \end{aligned} \tag{9.7}$$

Then $\left(\frac{1}{2} - \mu\right)\log n - \frac{1}{2}\log m \to \infty,\ \omega_n \log n \to \infty$ (see Section 8). Therefore (9.7) has the bound

$$B \exp(-q\rho_7(n)). \tag{9.8}$$

Summing this bound with respect to $3 \leq q \leq m$, we find

$$\frac{B}{\rho_8(n)}, \tag{9.9}$$

which proves the integral theorem for the zone $[1, \sqrt{m}]$. The zone $[-n^{\alpha}/\rho_6(n) - 1]$ is treated analogously, while the zone [-1, 1] corresponds to a known theorem. Thereby, Theorem 2 is proven.

10. Now let us consider Theorems 3, 4 and 5. They concern the narrowest zones. The functions h(x) of class III (see (1.5) [16]), which satisfy the condition

$$3 \log x \leqslant h(x) \leqslant M \log x, \tag{10.1}$$

where M is a constant, correspond to the case of the existence of a third moment, but, generally speaking, to the non-existence of moments, starting from some number. In this case, by classical means (see (1)), it can be established that $[0, \sqrt{\log n}/\rho(n)]$ and $[-\sqrt{\log n}/\rho(n), 0]$ will be z. u. l. n. a. for values of class (d), and z. n. a. in general, and that the zones $[0, \sqrt{\log n}\, \rho(n)]$; $[-\sqrt{\log n}\, \rho(n), 0]$ will not be such if not all moments exist. This is the substance of Theorem 5. These same results can be obtained by using the means described below.

Let us take Theorems 3 and 4, which pertain to functions of class II. For these functions we have (see Section 1 [16])

$$\rho_0(x) \log x \leqslant h(x) < (\log x)^2, \tag{10.2}$$

where

$$h(x) = M(x) \log x = N(\log x) \log x, \tag{10.3}$$

where

$$N'(z) \to 0 \text{ as } z \to \infty. \tag{10.4}$$

Let

$$\Lambda(n) = \sqrt{M(n)}\sqrt{\log n}. \tag{10.5}$$

If the zones $[0, \Lambda(n), \rho(n)]$ and $[-\Lambda(n)\rho(n), 0]$ are z. u. l. n. a. (it is understood that we are dealing with values of $X_j \in (d)$ or z. n. a.), then (0.1) must be fulfilled. This is proven in the same way as the corresponding assertion in Section 2. Now let us prove that condition (0.1) is sufficient in order that the zones $[0, \Lambda(n)/\rho(n)]$ and $[-\Lambda(n)/\rho(n), 0]$ be z. u. l. n. a.

Let $\mu > 0$ be a positive number, which will be fixed later on; $X_j \in (d)$, we have

$$p_{z_n}(x) = \frac{\sqrt{n}}{2\pi} \int_{-n^{-\mu}}^{n^{-\mu}} (\varphi(t))^n \exp(-itx\sqrt{n})\, dt + B \exp(-c_8 n^{1-2\mu}). \tag{10.6}$$

Following Section 4, let us take the bound

$$\varphi^{(q)}(t) = B \int_1^\infty x^q \exp(-h(x))\, dx = B \int_1^\infty \exp(q \log x - h(x))\, dx. \tag{10.7}$$

Let Q(q) be a solution of the equation

$$h'(x) = q + 4\log x, \quad \text{i.e.} \quad M(x) = q + 4. \tag{10.8}$$

Then

$$\int_{Q(q)}^{\infty} \exp(q\log x - h(x))\,dx = B\int_{Q(q)}^{\infty}\frac{dx}{x^4} = B, \tag{10.9}$$

$$\int_{1}^{\infty}\exp(q\log x - h(x))\,dx = BQ(q)\exp(q\log Q(q)) = B\exp((q+1)\log\varphi(q)). \tag{10.10}$$

Let us use this rough bound for $\varphi^{(q)}(t)$, which is sufficient for our purposes. From (10.8) we find

$$Q(q) = M^{-1}(q+4), \tag{10.11}$$

so that (10.10) gives

$$\varphi^{(q)}(t) = B\exp(q+1)\log M^{-1}(q+4). \tag{10.12}$$

Following Section 4 we find

$$K^{(q)}(0) = B\exp(q+1)\log M^{-1}(q+4), \tag{10.13}$$

$$\sup_{|t|<n^{-\mu}}\left|K^{(m)}(t)\frac{t^m}{m!}\right| = B\exp(\log M^{-1}(m+4) - \mu m\log n). \tag{10.14}$$

11. Now let us choose μ according to the condition

$$n^{1-2\mu} = (\Lambda(n))^2 = M(n)\log n. \tag{11.1}$$

Thus

$$\mu = \frac{1}{2} - \frac{X(n)}{\log n} + \frac{B}{\log n}; \quad X(n) = \log\Lambda(n) = O(\log\log n). \tag{11.2}$$

Let $\tau = 10^{-6}$. Let us take $\underline{m}$ under the condition that

$$\log M^{-1}(m+4) = \frac{\tau\mu\log n}{2} + B. \tag{11.3}$$

Then, on the strength of (10.4),

$$m + 4 = M(n^{\frac{\tau\mu}{2}}) = N\left(\frac{\tau\mu}{2}\log n\right) \asymp N(\log n) = M(n). \tag{11.4}$$

Now from formula (10.14) we obtain, according to (11.4), (11.2),

$$\sup_{|t|<n^{-\mu}}\left|K^{(m)}(t)\frac{t^m}{m!}\right| = B\exp\left(-\left(\mu - \frac{\tau}{2}\right)\log n + B\right)m = \tag{11.5}$$

$$= B\exp(-c_8 M(n)\log n) = B\exp(-c_8(\Lambda(n))^2).$$

Thus we have

$$p_{z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-n^{-\mu}}^{n^{-\mu}} \exp\left(-\frac{nt^2}{2} + nK_3(t) - i\sqrt{n}\,tx\right)dt + B\exp(-c_8(\Lambda(n))^2), \tag{11.6}$$

$$K_3(t) = \sum_{r=3}^{m} \psi_r \frac{t^r}{r!}. \tag{11.7}$$

Now we must deal with the integral function exp $nK_3(t)$. Let

$$\mu_1 = 0{,}99\ \mu, \tag{11.8}$$

and let us bound $nK_3(t)$ when $|t| \leq n^{-\mu_1}$. For $r \leq c_3$ we have $\psi_r = B$ and

$$n\sum_{3<r<c_3} \frac{t^r}{r!}\psi_r = B\frac{n}{n^{3\mu_1}} = O(1). \tag{11.9}$$

Then, when $c_3 < r \leq m$ we have, on the strength of (10.13),

$$\begin{aligned}\frac{\psi_r t^r}{r!} &= B\exp((r+1)\log M^{-1}(r+4) - \mu_1 r\log n - 2\log r) = \\ &= B\exp\left(-r\frac{\mu_1\log n}{10}\right) = Bn^{-\frac{3\mu_1}{10}}.\end{aligned} \tag{11.10}$$

From (11.9) and (11.10) at $|t| \leq n^{-\mu_1}$ we derive

$$nK_3(t) = B \tag{11.11}$$

and hence, using the Cauchy integral (see (7.2)),

$$\chi_r = B\,r!\,n^{r\mu_1}. \tag{11.12}$$

When $r \geq m$, $|t| \leq m^{-\mu_1}$, from (11.10) we find

$$\begin{aligned}\sum_{r=m}^{\infty} \frac{\chi_r t^r}{2!} &= B\exp(-0{,}005\, m\log n) = \\ &= B\exp(-c_9 M(n)\log n) = B\exp(-c_9(\Lambda(n))^2).\end{aligned} \tag{11.13}$$

12. Hence

$$\begin{aligned}p_{z_n}(x) = \frac{\sqrt{n}}{2\pi}\int_{-n^{-\mu}}^{n^{-\mu}} \exp\left(-\frac{nt^2}{2} - itx\sqrt{n}\right)\left(1 + \sum_{r=3}^{m}\frac{\chi_r}{r!}t^r\right)dt + \\ + B\exp(-c_{10}(\Lambda(n))^2).\end{aligned} \tag{12.1}$$

Let us substitute $\xi = t\sqrt{n}$:

$$\begin{aligned}p_{z_n}(x) = \frac{1}{2\pi}\int_{-n^{1/2-\mu}}^{n^{1/2-\mu}} \exp\left(-\frac{\xi^2}{2} - i\xi x\right)\left(1 + \sum_{r=3}^{m}\frac{\chi_r}{r!}\left(\frac{\xi}{\sqrt{n}}\right)^r\right)d\xi + \\ + B\exp(-c_{10}(\Lambda(n))^2).\end{aligned} \tag{12.2}$$

At a given $r \leq m$ let us consider

$$\frac{\chi_r}{r!\,(\sqrt{n})^r}\int_{n^{1/2-\mu}}^{\infty} \exp\left(-\frac{\xi^2}{2}\right)\xi^r d\xi = B^r\Gamma\left(\frac{r}{2}\right) n^{-\left(\frac{1}{2}-\mu_1\right)r} \exp\left(-\frac{1}{4}n^{1-2\mu}\right). \tag{12.3}$$

Factoring out $\exp\left(-\frac{1}{4}n^{1-2\mu}\right)$, let us bound the second factor in (12.3). Taking (11.4) into account, we find

$$B^r\Gamma\left(\frac{r}{2}\right)n^{-\left(\frac{1}{2}-\mu_1\right)r} = B\exp r\left(B+\frac{\log r}{2}-\left(\frac{1}{2}-\mu_1\right)\log n\right), \tag{12.4}$$

$$\log r-\left(\frac{1}{2}-\mu_1\right)\log n \leqslant \frac{1}{2}\log m-\left(\frac{1}{2}-\mu_1\right)\log n \leqslant -\frac{1}{4}\log n. \tag{12.5}$$

Therefore, (12.3) has the bound $Bn^{-r/4}$, and the sum with respect to $r > c_3$ gives (1). Then $n^{1-2\mu} = (\Lambda(n))^2$, so that from (12.2) we find

$$p_{z_n}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left(-\frac{\xi^2}{2}-i\xi x\right)\left(1+\sum_{r=3}^{m}\frac{\chi_r}{r!}\left(\frac{\xi}{\sqrt{n}}\right)^r d\xi + \right. \tag{12.6}$$

$$+B\exp(-c_{10}(\Lambda(n))^2);$$

the term $\frac{1}{\sqrt{2\pi}}e^{-x^2/2}$ is isolated from (12.6); here

$$0 \leqslant x \leqslant \frac{n^{1/2-\mu}}{\rho_7(n)} = \frac{\Lambda(n)}{\rho_7(n)}. \tag{12.7}$$

When $3 \leq r \leq c_3$ the sum of the corresponding terms will equal

$$o(1)e^{-x^2/2}. \tag{12.8}$$

When $c_3 < r \leq m$, following Section 8, we arrive at an examination of the expression

$$r(B-(1-2\rho)\log\rho_7(n)+\rho\log r-\rho(1-2\mu)\log n-(\mu-\mu_1)\log n). \tag{12.9}$$

Here $\log r \leq \log m = B\log\log n$ (see (11.4)). In view of this, (12.9) gives the bound

$$Bn^{-r(\mu-\mu_1)}. \tag{12.10}$$

Summation with respect to $c_3 \leq r \leq m$ gives the error

$$\frac{B}{n^2}; \tag{12.11}$$

after which, it follows from (12.6) that

$$p_{z_n}(x) = \frac{1}{\sqrt{2\pi}}\exp(-x^2/2)(1+o(1)). \tag{12.12}$$

The zone $[-\Lambda(n)/\rho_7(n), 0]$. is treated analogously.

13. The integral theorem is proven exactly as in Section 9; here there are rather rough bounds derived in Sections 10 through 12. It is also essential (see Section 9) that $(\Lambda(n))^2 = M(n)\log n \geqslant \rho_0(n)\log n$ (see (10.2)). In view of this

$$n^{k+1} \exp(-c_{11}(\Lambda(n))^2) = \exp(-c_{12}(\Lambda(n))^2) \tag{13.1}$$

(see formula (12.5) [16] and we can reason further, as in Section 9.

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